

# Phase-noise induced limitations on cooling and coherent evolution in opto-mechanical systems

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(Dated: October 15, 2009)*

We present a detailed theoretical discussion of the effects of ubiquitous laser noise on cooling and the coherent dynamics in opto-mechanical systems. Phase fluctuations of the driving laser induce modulations of the linearized opto-mechanical coupling as well as a fluctuating force on the mirror due to variations of the mean cavity intensity. We first evaluate the influence of both effects on cavity cooling and find that for a small laser linewidth the dominant heating mechanism arises from intensity fluctuations. The resulting limit on the final occupation number scales linearly with the cavity intensity both under weak and strong coupling conditions. For the strong coupling regime, we also determine the effect of phase noise on the coherent transfer of single excitations between the cavity and the mechanical resonator and obtain a similar conclusion. Our results show that conditions for optical ground state cooling and coherent operations are experimentally feasible and thus laser phase noise does pose a challenge but not a stringent limitation for opto-mechanical systems.

PACS numbers: 42.50.Lc, 42.50.Wk, 07.10.Cm

Over the past years tremendous experimental progress with opto-mechanical devices [1, 2] and analogous systems in the microwave regime [3, 4] has been made. Many groups have by now achieved significant cooling of mechanical motion [5, 6, 7, 8, 9, 10, 11] and in several of these setups resolved-sideband conditions have been demonstrated, which is a necessary prerequisite for quantum ground state cooling [12, 13, 14, 15]. Beyond laser cooling, which works well in the regime of weak opto-mechanical coupling, the recently demonstrated strong coupling regime [16] might allow for the observation of coherent dynamics between the cavity field and the vibrating mirror. For example, the transfer of single photonic excitations to the phonon mode has been suggested [17, 18] to prepare and study quantum superpositions of macroscopic objects.

Despite a steady experimental progress cooling to the ground state and further the combination of strong coupling conditions with low occupation numbers have not been achieved so far. The main limitation in current systems arises from the re-thermalization rate of the mirror  $\Gamma_m = k_B T / \hbar Q_m$ , (where  $Q_m$  is the mechanical quality factor and  $T$  the temperature of the support), which competes with optical cooling. However, with lower base temperatures and increasing mechanical quality factors, mechanical heating can be strongly reduced and the impact of other imperfections on opto-mechanical systems must be considered. In particular, it has been argued recently [19] that ubiquitous laser phase fluctuations impose severe limitations on opto-mechanical cool-

ing schemes and can impede ground state cooling under current experimental parameters. While effects of laser noise have indeed been observed [9], the laser linewidth requirements  $\Gamma_L \approx 10^{-3}$  Hz for ground state cooling estimated in [19] are clearly inconsistent with current experimental achievements where residual mean occupancies as low as  $n_0 \approx 30$  [20, 21, 22] have been achieved with  $\Gamma_L \sim 1$  kHz. This discrepancy can be understood from a suppression of noise at the mechanical frequency  $\omega_m$  [9], but a rigorous derivation has not been given so far. Furthermore, potential impairing effects of a finite laser linewidth on strongly coupled opto-mechanical systems have not been addressed yet.

Given the great interest in opto-mechanical experiments the impact of phase noise in such systems deserves a thorough theoretical investigation, which we provide in this work. We here generalize the standard descriptions of opto-mechanical systems to include laser phase noise with arbitrary frequency noise spectrum  $S_\phi(\Omega)$  and evaluate its influence on cooling and coherent oscillations. The main results of this work are as follows. In accordance with previous predictions [9, 19] we obtain a lower limit for the final mirror occupation number which is proportional to the intensity and the noise spectrum  $S_\phi(\omega_m)$  at the mechanical resonance frequency  $\omega_m$ . Surprisingly, this result applies for both the weak and strong coupling regime and we find that for a given  $\Gamma_m$  and  $g_0$  (opto-mechanical coupling *per single photon*), the condition

$$S_\phi(\omega_m) < g_0^2 / \Gamma_m, \quad (1)$$

must be satisfied in order to achieve ground state cooling. In the strong coupling regime we derive a similar condition for the observation of coherent oscillations,

$$S_\phi(\omega_m) < g_0^2 / \kappa, \quad (2)$$

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where  $\kappa$  is the cavity field decay rate. Importantly, we also show that under relevant conditions decoherence from low frequency phase noise in opto-mechanical systems is negligible. Thus a small laser linewidth  $\Gamma_L/\kappa \ll 1$  together with suppression of phase noise at high frequencies is sufficient to enable ground state cooling and the observation of coherent oscillations. The conditions (1) and (2) are experimentally challenging, but well within reach with state of the art laser stabilization.

In the next section we develop the model for opto-mechanical coupling including laser noise, closely following the lines of [13, 23]. The results on weak and strong coupling are derived in Sec. II and Sec. III, respectively. Details of calculations are moved to three appendices.

## I. MODEL

We consider a typical opto-mechanical setup of an optical cavity mode coupled to a micro- or nanomechanical oscillator. We will refer in the following to the prototype example of an opto-mechanical system, a Fabry-Perot cavity of length  $L$  with one fixed, heavy mirror (which serves as the input coupler) and a light vibrating micro-mirror. However, our results apply as well to microtoroidal cavities or to dielectric membranes in a Fabry-Perot cavity. We restrict our analysis to a single vibrational mode of the mirror of mass  $m$  and frequency  $\omega_m$  and a single cavity mode of frequency  $\omega_c$  that is driven close to resonance by a laser of frequency  $\omega_l$ . According to [24] the field couples to the motion of the mirror via the radiation pressure interaction and a total Hamiltonian can be written (in a frame rotating at  $\omega_l$ )

$$H = -\Delta_0 a^\dagger a + \omega_m b^\dagger b + g_0 a^\dagger a (b + b^\dagger) - i \left( \mathcal{E}^*(t) a - \mathcal{E}(t) a^\dagger \right). \quad (3)$$

Here  $a$ ,  $a^\dagger$  are the bosonic operators for the cavity mode such that  $[a, a^\dagger] = 1$  and  $b$ ,  $b^\dagger$  for the mechanical mode  $[b, b^\dagger] = 1$ , while  $\Delta_0 = \omega_l - \omega_c$ . The optomechanical coupling is  $g_0 = (\omega_c/L) \sqrt{\hbar/(2m\omega_m)}$  and  $\mathcal{E}(t)$  is the noisy laser field with an absolute value  $|\mathcal{E}(t)| \sim \mathcal{E}_0$ . While our analysis can be generalized to arbitrary noise processes we focus in the following solely on phase noise,

$$\mathcal{E}(t) = \mathcal{E}_0 e^{i\phi(t)}. \quad (4)$$

Here  $\phi(t)$  is a Gaussian noise process with zero mean which is characterized by the correlation function

$$\langle \dot{\phi}(s) \dot{\phi}(s') \rangle_{cl} = \int \frac{d\Omega}{2\pi} S_{\dot{\phi}}(\Omega) e^{-i\Omega(s-s')}, \quad (5)$$

where  $\{O\}_{cl}$  denotes the average over different noise realizations. The frequency noise spectrum  $S_{\dot{\phi}}(\Omega)$  is specific for each experimental setup. For concreteness we will consider below a simplified noise model with a noise spectrum and correlation function,

$$S_{\dot{\phi}}(\Omega) = \frac{2\Gamma_L \gamma_c^2}{\gamma_c^2 + \Omega^2}, \quad \langle \dot{\phi}(s) \dot{\phi}(s') \rangle_{cl} = \Gamma_L \gamma_c e^{-\gamma_c |s-s'|}, \quad (6)$$

respectively. Here  $\Gamma_L$  can be identified with the laser linewidth which is typically known. The additional parameter  $\gamma_c^{-1}$  characterizes a finite correlation time of the underlying noise process and leads to a suppression of noise at high frequencies. In the limit  $\gamma_c \rightarrow \infty$  we obtain the  $\delta$ -correlated noise model used in [19].

In addition to the classical driving field, the cavity field is coupled to the electromagnetic modes of the environment. We describe the resulting dissipative dynamics by a master equation

$$\dot{\rho} = -i[H, \rho] + \mathcal{L}_f(\rho), \quad (7)$$

where  $\mathcal{L}_f(\rho) = \kappa(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a)$  and  $\kappa$  is the cavity field decay rate. We made use of the fact that for optical experiments even at room temperature the thermal photon number is effectively zero [25]. In principle Eq. (7) should also contain the term describing coupling of the mechanical system to its thermal environment. The influence of thermal noise on opto-mechanical cooling schemes has been studied in previous works [12, 13, 14, 15]. To focus on the effects of laser noise only we here consider the regime of efficient laser cooling where thermal noise is already suppressed and the dynamics is well described by the master equation (7).

### Linearized opto-mechanical coupling

The radiation pressure from the cavity field leads to a mean displacement of the mirror and we are only interested in fluctuations around this shifted equilibrium position. We therefore perform a unitary transformation  $b \rightarrow \beta + b$  where  $\beta = -g_0 n_{ph}/\omega_m$  is the mean displacement amplitude and  $n_{ph} = \{\langle a^\dagger a \rangle_q\}_{cl}$  is the mean cavity photon number averaged over both quantum and classical fluctuations. In addition, we perform another unitary displacement operation for the intracavity field  $a \rightarrow \alpha(t) + a$  where

$$\alpha(t) = \int_{-\infty}^t ds e^{-(i\Delta + \kappa)(t-s)} \mathcal{E}(s), \quad (8)$$

is the classical part of the stochastically driven cavity field and  $\Delta = \Delta_0 - 2g_0\beta$  is the effective detuning. Note that by choosing this time-dependent displacement amplitude  $\alpha(t)$  instead of a constant one [19] we can unambiguously interpret  $a$  as the quantum field of a non-driven cavity.

From these transformations we obtain the master equation

$$\dot{\rho} = -i[H_{opt}(t) + g_0 \mathcal{N}(t)(b + b^\dagger), \rho] + \mathcal{L}_f(\rho), \quad (9)$$

where  $\mathcal{N}(t) = |\alpha(t)|^2 - \langle |\alpha(t)|^2 \rangle_{cl}$  measures intensity fluctuations of the mean intracavity field and the opto-mechanical is now given by

$$H_{opt}(t) = \omega_m b^\dagger b - \Delta a^\dagger a + g_0 \left( \alpha^*(t) a + \alpha(t) a^\dagger \right) (b + b^\dagger). \quad (10)$$

Note that in Eq. (10) we have already neglected the nonlinear term  $\sim a^\dagger a$  which is equivalent to an expansion in  $1/\sqrt{n_{ph}}$  and well justified by current experimental setups.

From the equation of motion (9) we conclude that laser noise contributes two effects: First, it causes a modulation of the opto-mechanical coupling strength  $\sim \alpha(t)$ . Second, it induces a stochastic force on the mirror  $\sim N(t)$ , which arises from a conversion of phase to amplitude fluctuations inside the optical resonator. This heating mechanism has no direct analog in laser cooling in free space but a similar effect has been observed with trapped atoms inside a cavity [26, 27]. In the following we evaluate the influence of both contributions on cavity cooling and the coherent evolution in this system.

## II. WEAK COUPLING REGIME

We first consider the weak coupling limit  $g_0|\alpha(t)| \ll \kappa$ . In this regime the cavity mode acts as a dissipative channel for the resonator mode which for example can be exploited for cooling. We are interested in the effects of phase noise on the mean mirror occupation number in steady state  $n_0 = \{\langle b^\dagger b \rangle_q\}_{cl}(t \rightarrow \infty)$ . From the linearized Hamiltonian (10) we can derive a set of coupled differential equations for the quantum averages, e.g.  $\langle b^\dagger b \rangle_q$ , which can be solved in the perturbative limit  $g_0|\alpha(t)| \ll \kappa$ . Averaging the result over the classical noise process [28] we obtain an effective cooling equation

$$\dot{n} = -W(n - n_0), \quad (11)$$

with  $n = \{\langle b^\dagger b \rangle_q\}_{cl}$ . Here the cooling rate  $W = S(\omega_m) - S(-\omega_m)$  and the average steady state occupation number  $n_0 = S(-\omega_m)/W$  depend on the total noise spectrum  $S(\omega) = S_N(\omega) + S_A(\omega)$  which is the sum of the modulated intracavity amplitude correlation spectrum

$$S_A(\omega) = 2g_0^2 \text{Re} \int_0^\infty d\tau \{\alpha^*(\tau)\alpha(0)\}_{cl} e^{i\Delta\tau} e^{-\kappa\tau} e^{i\omega\tau}, \quad (12)$$

and the spectrum of intensity fluctuations,

$$S_N(\omega) = 2g_0^2 \text{Re} \int_0^\infty d\tau \{N(\tau)N(0)\}_{cl} e^{i\omega\tau}. \quad (13)$$

For the case of a non-fluctuating driving field  $\mathcal{E}(t) = \mathcal{E}_0$ ,  $\alpha(\tau) = \alpha_0 = \mathcal{E}_0/(i\Delta + \kappa)$  and  $S_N(\omega) = 0$ . Then, by evaluating Eq. (12) for the optimized detuning  $\Delta = \Delta_{op} = -\sqrt{\kappa^2 + \omega_m^2}$  we recover the standard results,

$$W_0 = \frac{2g_0^2|\alpha_0|^2}{\kappa} \left( \frac{\omega_m}{|\Delta_{op}|} \right), \quad n_0 = \frac{1}{2} \left( \frac{|\Delta_{op}|}{\omega_m} - 1 \right), \quad (14)$$

which are well known from the theory of opto-mechanical laser cooling [12, 13]. In particular in the sideband resolved regime  $\omega_m \gg \kappa$  the minimal occupation number is  $n_0 = \kappa^2/4\omega_m^2$ .

Our goal in the following is to study the influence of a noisy driving field on the final temperature. From the definition of  $\alpha(t)$  given in Eq. (8) and  $S_A(\omega)$ ,  $S_N(\omega)$  given in Eqs. (12) and (13) we see that the final occupation number  $n_0$  depends on integrals over two and four point correlation functions of

the stochastic quantity  $e^{i\phi(t)}$ , which itself is a non-linear function of the noise process  $\dot{\phi}(t)$ . For the remainder of the paper we therefore assume that the total change in phase  $\phi(t)$  accumulated on a timescale of  $\kappa$  is small. This assumption is equivalent to the experimentally relevant limit  $\Gamma_L \ll \kappa$  and allows a rigorous expansion of noise correlation functions in powers  $\Gamma_L/\kappa$ . The details of these calculations are shifted to App. A and B and here we summarize only the main results.

### Amplitude fluctuations

Let us first look at phase noise induced modifications of  $S_A(\omega)$ . To provide some physical insight we can assume in a zeroth order approximation that the cavity field simply follows the driving field adiabatically,

$$\alpha(t) \approx \mathcal{E}_0 e^{i\phi(t)} / (\kappa - i\Delta). \quad (15)$$

By inserting this expression into the definition of  $S_A(\omega)$  we obtain

$$S_A(\omega)/W_0 \simeq \kappa \text{Re} \int_0^\infty d\tau \{e^{i\phi(\tau)} e^{-i\phi(0)}\}_{cl} e^{i(\Delta+\omega)\tau} e^{-\kappa\tau}, \quad (16)$$

and a simple result for this integral can be found in the white noise limit  $\gamma_c \rightarrow \infty$  where  $\{e^{i\phi(\tau)} e^{-i\phi(0)}\}_{cl} \simeq e^{-\Gamma_L|\tau|}$ . From this estimate we expect correction to  $S_A(\pm\omega_m)$  and therefore to the final occupation number  $n_0$  which are of order  $O(\Gamma_L/\kappa)$ .

In App. A we present a more rigorous calculation of  $S_A(\omega)$  for  $\Gamma_L/\kappa \ll 1$  and here briefly discuss results for the resolved sideband regime  $\kappa \ll \omega_m$ . For transitions on the red sideband, which are associated with a resonant exchange of a vibrational quanta and a cavity photon, we obtain

$$\frac{S_A(\omega_m)}{W_0} \simeq 1 - \int \frac{d\Omega}{2\pi} \frac{S_\phi(\Omega)}{\kappa^2 + \Omega^2} \geq 1 - \frac{\Gamma_L}{\kappa}. \quad (17)$$

Here we obtain a dependence on the low frequency part of the noise spectrum and the lower bound which has been derived using the model defined in Eq. (6) agrees with the estimates from above. For the blue sideband transitions, which correspond to a simultaneous excitations of a photon and a phonon we obtain instead

$$\frac{S_A(-\omega_m)}{W_0} \simeq \frac{\kappa^2}{4\omega_m^2} + \frac{\kappa}{18\omega_m^2} \left[ S_\phi(\omega_m) + \frac{1}{4} S_\phi(2\omega_m) \right], \quad (18)$$

i.e., a modification which depends on phase noise at high frequencies. The contribution at  $2\omega_m$  can be understood from the fact that for  $\Delta = -\omega_m$  it takes this amount of energy to excite a photon and a phonon. Although noise at  $\omega_m$  is non-resonant with this excitation process it is maximally enhanced by the cavity response, and therefore leads to an equivalent contribution. However, in both cases the noise is either non-resonantly driving the system or is suppressed by the cavity response and the resulting corrections to the final occupation number only scale as  $\sim \kappa^2/\omega_m^2 \times \Gamma_L/\kappa$ , i.e. they are reduced by the sideband parameter  $\kappa/\omega_m$ . In summary we conclude that phase modulations of the cavity field lead to a reduction of the cooling rate by  $\Gamma_L/\kappa$ , the ratio of laser to cavity linewidth, and therefore pose no serious experimental limitation.

### Intensity fluctuations

We now look at intensity fluctuations. As the zeroth order approximation of  $\alpha(t)$  would lead to a constant intensity  $|\alpha(t)|^2$  we expect the first correction for a slow noise process to scale as  $\sim \dot{\phi}(t)/\kappa$  and we therefore approximate

$$\alpha(t) \approx \frac{\mathcal{E}_0 e^{i\phi(t)}}{\kappa - i\Delta} \left( 1 - i \frac{\dot{\phi}(t)}{\kappa - i\Delta} \right). \quad (19)$$

Inserting this expression into the definition of  $S_N(\omega)$  in Eq. (13) we obtain for  $\Delta = -\omega_m$ ,

$$\frac{S_N(\omega_m)}{W_0} \sim |\alpha_0|^2 \text{Re} \int_0^\infty \{\dot{\phi}(\tau)\dot{\phi}(0)\}_{cl} e^{i\omega_m \tau} d\tau. \quad (20)$$

Already from this simple estimate we find that in accordance with earlier predictions [9, 19] intensity fluctuations do impose a limit on the final occupation number which increases with the mean cavity photon number  $|\alpha_0|^2$ . However, Eq. (20) also predicts a crucial dependence of this limit on the particularities of the noise process, i. e., the correlation function  $\{\dot{\phi}(\tau)\dot{\phi}(0)\}_{cl}$ . This means that the final occupation number depends in a non-universal way on the phase noise characteristics and cannot be inferred from a white noise model described by the linewidth only.

The basic prediction of Eq. (20) is confirmed by a more rigorous derivation outlined in App. B where we evaluate the correlation function  $\{N(\tau)N(0)\}_{cl}$  to first order in the parameter  $\Gamma_L/\kappa$ . Under this assumption and  $\Delta = \Delta_{op}$  the resulting limit on the final occupation number can be written as

$$n_0 \geq \frac{S_N(\omega_m)}{W} \simeq |\alpha_0|^2 \frac{S_\phi(\omega_m)}{2\kappa} \left( \frac{|\Delta_{op}|}{\omega_m} \right). \quad (21)$$

This is the main result on laser noise induced limitations of opto-mechanical cooling in the weak coupling regime. It agrees with a simplified analysis given in Ref. [9] for the resolved sideband regime and generalizes the result for the white noise limit derived in Ref. [19] for arbitrary noise processes. It is instructive to consider the toy model for laser noise as given in Eq. (6) from which we obtain in the sideband resolved regime

$$n_0 \geq |\alpha_0|^2 \frac{\Gamma_L}{\kappa} \frac{\gamma_c^2}{\gamma_c^2 + \omega_m^2}. \quad (22)$$

For typical experimental parameters  $|\alpha_0|^2 \approx 10^{10}$ ,  $\omega_m \approx 10$  MHz,  $\kappa \approx 1$  MHz and  $\Gamma_L \approx 1$  kHz the assumption of a white noise model,  $\gamma_c \rightarrow \infty$ , would lead to the prediction  $n_0 \gtrsim 10^5$ , which is in sharp contrast with observed experimental data. However, for a realistic noise model with a finite cutoff frequency  $\gamma_c \ll \omega_m$  intensity fluctuations are strongly suppressed and the resulting limits  $n_0 \geq 1 - 100$  are consistent with recent experiments.

It was pointed out already in [9] that the scaling of the bound in (21) with the intracavity photon number  $|\alpha_0|^2$  implies an optimal driving power balancing the cooling effects with heating due to laser noise. If we combine the present result

with the known bounds from the theory of opto-mechanical cooling with an ideal laser [12, 13] we get

$$n_0 \simeq \frac{2\kappa\Gamma_m}{g_0^2|\alpha_0|^2} + \frac{\kappa^2}{4\omega_m^2} + \frac{|\alpha_0|^2}{2\kappa} S_\phi(\omega_m). \quad (23)$$

Here  $\Gamma_m = k_B T / \hbar Q_m$  is the mechanical heating rate for a resonator with quality factor  $Q_m$  coupled to a thermal phonon reservoir of temperature  $T$ . The first term on the right hand side is the residual thermal occupation, the middle term is the contribution from heating due to Stokes scattering, and the last term stems from laser noise. For an optimal choice of the intracavity photon number  $|\alpha_0|^2 = (2\kappa)^2 \Gamma_m / g_0^2 S_\phi(\omega_m)$  we obtain

$$n_0 \simeq 2 \sqrt{\frac{\Gamma_m S_\phi(\omega_m)}{g_0^2}} + \frac{\kappa^2}{4\omega_m^2}. \quad (24)$$

From this result we identify Eq. (1) as the relevant condition to achieve ground state cooling in weakly coupled opto-mechanical systems, provided the optimal value of  $|\alpha_0|^2$  is not prohibitive. Note that this condition depends on  $g_0$ , the radiation pressure coupling per *single photon*, cf. Eq. (3).

The details of the laser noise characteristics will depend on the concrete experimental setup, but the spectrum of intensity fluctuations  $S_N(\omega)$  can in each case be directly measured, e.g., from correlations of the transmitted intensity  $I_{out}(t)$ . In the strongly driven regime the cavity out-field is dominated by the classical part  $b_{out}(t) \sim \sqrt{\kappa} \alpha(t)$  and using standard results on photon counting statistics [29] we obtain the simple relation

$$\frac{S_N(\omega)}{W_0} = \frac{1}{2} \left[ \frac{S_I(\omega)}{S_{sn}} - 1 \right]. \quad (25)$$

Here  $S_I(\omega) = \int_{-\infty}^\infty d\tau C_I(\tau) e^{i\omega\tau}$  is the spectrum of the normalized photon current correlation function  $C_I(\tau) = \{I_{out}(\tau)I_{out}(0)\}_{cl} / \{I_{out}(0)\}_{cl}^2 - 1$  and  $S_{sn}$  is the shot noise contribution thereof.

### III. STRONG COUPLING REGIME

We now consider an opto-mechanical system operated in the strong coupling regime where the linear photon-phonon interaction strength  $G = g_0 \alpha_0$  exceeds the cavity linewidth  $\kappa$ . This regime has been recently studied experimentally [16] and has been discussed theoretically in the context of opto-mechanical cooling [23, 30]. However, more importantly strong coupling conditions enable a coherent exchange of photonic and mechanical excitations. Thus, the opto-mechanical system can serve as a quantum interface between photons and phonons with potential applications for state preparation and quantum measurements of macroscopic mechanical motion. It is therefore worthwhile to study the role of phase noise in particular under strong coupling conditions where due to the required large values of  $|\alpha_0|$  more pronounced effects are expected.



### Strong coupling

Let us first briefly review the main features of the opto-mechanical system in the strong coupling regime, ignoring for the moment the presence of phase noise or other imperfections. When the interaction between the cavity and the resonator mode exceeds the cavity decay rate the system dynamics is conveniently described in terms of the new collective operators  $A_{\pm}$  which diagonalize  $H_{\text{opt}}$ ,

$$H_{\text{opt}} = \omega_+ A_+^\dagger A_+ + \omega_- A_-^\dagger A_- . \quad (26)$$

Assuming resonance conditions  $\Delta_c = -\omega_m$  and  $|G| < \omega_m/2$  which is required for stability [14], the eigenfrequencies are  $\omega_{\pm} = \omega_m(1 \pm 2|G|/\omega_m)^{1/2}$  and the normal modes are approximately given by

$$A_{\pm} \simeq \frac{e^{-i\theta}}{\sqrt{2}} b \pm \frac{e^{i\theta}}{\sqrt{2}} a \mp \frac{G^*}{2\omega_m} \left( \frac{e^{i\theta}}{\sqrt{2}} b^\dagger \pm \frac{e^{-i\theta}}{\sqrt{2}} a^\dagger \right), \quad (27)$$

where we have defined  $e^{i2\theta} := \alpha_0/|\alpha_0|$ . For not too large values of  $|G|$  the eigenmodes are essentially equal superpositions of the original cavity and resonator mode and they are split in frequency by  $\omega_+ - \omega_- \simeq 2|G|$ . A normal mode splitting exceeding the cavity decay rate  $\kappa$  is a first signature of the strong coupling regime and has recently been observed in experiments [16]. In this limit the Liouville operator is approximately given by [23]

$$\begin{aligned} \mathcal{L}_f(\rho) \simeq & \frac{\kappa}{2}(n_0 + 1) \sum_{\xi=\pm} \left( 2A_{\xi}\rho A_{\xi}^\dagger - A_{\xi}^\dagger A_{\xi}\rho - \rho A_{\xi}^\dagger A_{\xi} \right) \\ & + \frac{\kappa}{2}n_0 \sum_{\xi=\pm} \left( 2A_{\xi}^\dagger \rho A_{\xi} - A_{\xi} A_{\xi}^\dagger \rho - \rho A_{\xi} A_{\xi}^\dagger \right). \end{aligned} \quad (28)$$

As expected we see that both modes decay with half of the cavity decay rate. A lower limit on the achievable occupation numbers  $n_0 = |G|^2/4\omega_m^2 \ll 1$  arises from small admixtures of  $a^\dagger$  and  $b^\dagger$  in Eq. (27) due to energy non-conserving terms in  $H_{\text{opt}}$ . Neglecting this small correction we immediately see from Eq. (28) that without any additional heating mechanisms the total number of excitations in the system  $n_{\text{tot}} = \langle A_+^\dagger A_+ \rangle + \langle A_-^\dagger A_- \rangle$  decays as

$$\dot{n}_{\text{tot}} = -\kappa n_{\text{tot}}. \quad (29)$$

Therefore, in the strong coupling limit the opto-mechanical cooling rate  $W$  is independent of the driving strength and saturates at the maximum value set by the cavity field decay rate  $\kappa$ .

### Strong coupling in the presence of phase noise

In the presence of noise the picture above is modified on one hand by the presence of fluctuating forces and on the other hand by a modulated cavity-resonator coupling  $G \rightarrow G(t) := g_0\alpha(t)$ . To account for this time-dependent coupling we introduce time-dependent mode operators  $A_{\pm}(t)$  which diagonalize

the Hamiltonian  $H_{\text{opt}}(t)$  given in Eq. (10) at each point in time, i.e.

$$[H_{\text{opt}}(t), A_{\pm}(t)] = -\omega_{\pm}(t)A_{\pm}(t). \quad (30)$$

Here  $\omega_{\pm}(t) = \omega_m(1 \pm 2|G(t)|/\omega_m)^{1/2}$  are the instantaneous eigenfrequencies and the decomposition of  $A_{\pm}(t)$  contains now also time dependent phases  $e^{i2\theta(t)} := \alpha(t)/|\alpha(t)|$ . To simplify the following discussion and to focus on the effect of noise we assume that the condition  $\kappa \ll |G(t)| \ll \omega_m$  is strictly fulfilled, which allows us to neglect in Eq. (27) contributions of order  $O(G(t)/\omega_m)$ . Under this assumption

$$\dot{\rho} = -i[H(t), \rho] + \mathcal{L}_f(t)(\rho), \quad (31)$$

where

$$H(t) = \sum_{\xi=\pm} \omega_{\xi}(t) A_{\xi}^\dagger(t) A_{\xi}(t) + \frac{g_0}{\sqrt{2}} \left( e^{i\theta(t)} A_{\xi}(t) + e^{-i\theta(t)} A_{\xi}^\dagger(t) \right) \mathcal{N}(t), \quad (32)$$

and

$$\mathcal{L}_f(t)(\rho) \simeq \frac{\kappa}{2} \sum_{\xi=\pm} \left( 2A_{\xi}(t)\rho A_{\xi}^\dagger(t) - A_{\xi}^\dagger(t)A_{\xi}(t)\rho - \rho A_{\xi}^\dagger(t)A_{\xi}(t) \right). \quad (33)$$

We now investigate how those modifications affect cooling and coherent dynamics in the strong coupling regime.

### Cooling

We first look at phase noise induced limitations for ground state cooling in the strong coupling regime. In the previous discussion on the weak coupling regime we have seen that the effect of the modulation of  $\alpha(t)$  is less crucial for cooling than that of the fluctuations of the cavity intensity. We therefore neglect for the moment the explicit time dependence of  $A_{\pm}(t)$  and study the effect of  $\mathcal{N}(t)$  only. A justification for this approximation follows from the analysis present below. We obtain the coupled equations

$$\langle \dot{N}_{\pm} \rangle = -\kappa \langle N_{\pm} \rangle - ig_0 \mathcal{N}(t) \langle e^{-i\theta} A_{\pm}^\dagger - e^{i\theta} A_{\pm} \rangle, \quad (34)$$

$$\langle \dot{A}_{\pm} \rangle = -(i\omega_{\pm} + \kappa/2) \langle A_{\pm} \rangle - ig_0 \mathcal{N}(t) e^{-i\theta}, \quad (35)$$

where  $N_{\pm} = A_{\pm}^\dagger A_{\pm}$ . After averaging over the noise we end up with

$$\langle \dot{N}_{\pm} \rangle_{cl} = -\kappa \langle N_{\pm} \rangle_{cl} + S_N(\omega_{\pm} + i\kappa/2), \quad (36)$$

where  $S_N(\omega)$  is defined in Eq. (13). In the limit  $\Gamma_L/\kappa \ll 1$  we can use the results derived in App. B to evaluate this quantity. By assuming that for frequencies  $\Omega \sim \omega_m$  the noise spectrum  $S_{\phi}(\Omega)$  is flat on a scale  $\kappa$  and to lowest order in  $\kappa/G$  we obtain

$$S_N(\omega_{\pm} + i\kappa/2) \simeq g_0^2 |\alpha_0|^4 \left[ \frac{4\omega_m^2 S_{\phi}(\omega_{\pm})}{(\omega_{\pm}^2 - \omega_m^2)^2} + \frac{S_{\phi}(\omega_m)}{2(\omega_{\pm} - \omega_m)^2} \right]. \quad (37)$$

Interestingly, heating arises from two contributions. As expected, the first term represents intensity fluctuations at the eigenfrequency  $\omega_{\pm}$ . Since in the strong coupling regime these

frequencies are well separated from the cavity resonance, the noise at these frequencies is suppressed by  $\omega_m^2/(\omega_\pm^2 - \omega_m^2)^2$ . The second contribution in Eq. (37) represents noise which is resonantly enhanced by the cavity. Although it is not resonant with  $\omega_\pm$  it still couples to the damped motion of  $A_\pm$  and we see that both terms lead to similar contributions for heating. For  $\omega_\pm \simeq \omega_m \pm G$  the lower limit for the total steady state occupation number is then given by

$$n_{\text{tot}} \geq |\alpha_0|^2 \frac{S_\phi(\omega_m) + S_\phi(\omega_+) + S_\phi(\omega_-)}{\kappa}. \quad (38)$$

Surprisingly for  $S_\phi(\omega_m) \approx S_\phi(\omega_\pm)$  this result is quite similar to the weak coupling regime which we attribute to a cancellation of two effects. On one hand for increasing  $\alpha_0$  the cooling rate saturates in the strong coupling regime at  $W \simeq \kappa$ . One would therefore naively expect a scaling  $n_0 \sim |\alpha_0|^4$ . However, since the eigenfrequencies  $\omega_\pm$  are well detuned from the cavity resonance the effect of phase fluctuations is suppressed by  $1/G^2 \sim 1/g_0^2|\alpha_0|^2$  (see Eq. (37)) which reduces in the strong coupling limit the scaling from  $|\alpha_0|^4$  to  $|\alpha_0|^2$ .

Due to the saturation of the cooling rate the strong coupling regime does not offer a particular advantages for cooling. We therefore do not go further into details and study instead the effects of phase noise for coherent dynamics where strong coupling conditions are essential.

### Coherent oscillations

As already mentioned above, the interesting aspect about the strong coupling regime of opto-mechanical systems is the ability to realize a coherent interface between mechanical and optical modes. For example, if the resonator is initially prepared in the ground state (possibly in combination with other cooling methods) and the cavity in a Fock state  $|1\rangle_c$  this single excitation is swapped onto the resonator mode at time  $t_s = \pi/(2|G|)$  and recovered at a later time  $t_r = \pi/|G|$ . More general we can describe this process in terms of an arbitrary coherent state  $|\xi_c\rangle_c$  which in a frame rotating with  $\omega_r$  evolves under ideal conditions as

$$|0\rangle_r |\xi_c\rangle_c \rightarrow |-ie^{-i2\theta} \xi_c\rangle_r |0\rangle_c \rightarrow |0\rangle_r |-\xi_c\rangle_c. \quad (39)$$

Since this evolution is independent of the initial coherent state amplitude it can be generalized to arbitrary quantum states. Therefore, apart from a known phase the dynamics generated by  $H_{op}$  implements a faithful mapping between the states of the cavity and the resonator mode. If this operation is fast compared to photon loss, i.e.  $\kappa \ll |G|$ , it can be employed as a coherent way for an optical preparation of motional states. Alternatively the reverse process would enable an optical detection of the resonator state.

To characterize coherent oscillations in the presence of photon loss and phase noise it is sufficient to study the evolution of an initial coherent state  $|\psi_0\rangle = |\xi_r(0)\rangle_r |\xi_c(0)\rangle_c$  under the evolution of the effective Hamiltonian  $H_{\text{eff}}(t) = H(t) - i\kappa c^\dagger c$ . Ignoring small corrections of order  $G/\omega_m \ll 1$  which are not essential in the following discussion the state will then evolve

into  $|\psi(t)\rangle = |\xi_r(t)\rangle_r |\xi_c(t)\rangle_c$  where the corresponding coherent state amplitudes can be written as

$$\begin{pmatrix} \xi_r(t) \\ \xi_c(t) \end{pmatrix} = \begin{pmatrix} c_{bb}(t) & c_{ba}(t) \\ c_{ab}(t) & c_{aa}(t) \end{pmatrix} \begin{pmatrix} \xi_r(0) \\ \xi_c(0) \end{pmatrix} + \begin{pmatrix} c_b(t) \\ c_a(t) \end{pmatrix}. \quad (40)$$

The ideal evolution described in Eq. (39) suggest to use the state overlaps  $|\langle -i\xi_c(0)e^{-i2\theta}|\xi_c(t)\rangle|^2$  and  $|\langle -\xi_c(0)|\xi_c(t)\rangle|^2$  to characterize a state transfer or a full oscillation respectively. We here choose the latter option and for an distribution  $P(r, \theta)$  of initial coherent state amplitudes  $\xi_c(0) = re^{i\phi}$  we define the fidelity

$$\mathcal{F}(t) = \frac{1}{\pi} \int_0^\infty r dr \int_0^{2\pi} d\phi \{e^{-|\xi_c(t)+re^{i\phi}|^2}\}_{cl} P(r, \theta). \quad (41)$$

To be more concrete we average over initial states with  $r = 1$  and for  $\xi_r(0) = 0$  we finally end up with

$$\mathcal{F}(t) = e^{-|c_{aa}(t)+1|^2}_{cl} \times e^{-|c_a(t)|^2}_{cl}. \quad (42)$$

While there is some arbitrariness in this choice of a fidelity, the definition (42) is sensitive to different aspects of the noise and should therefore be a good characterization for coherent processes involving a low number of excitations. For a noiseless system  $c_a(t) = 0$  and we obtain

$$\mathcal{F}_0(t) = e^{-|1+\cos(|G|t)e^{-\kappa t/2}|^2}. \quad (43)$$

The fidelity for a full oscillation is approximately given by  $\mathcal{F}_0(t_r) \simeq e^{-\pi^2 \kappa^2 / |G|^2}$ .

To study the additional degrading of  $\mathcal{F}(t)$  in the presence of noise we write  $a(t) \simeq e^{-i\theta(t)}(A_+(t) - A_-(t))/\sqrt{2}$ . The coefficients  $c_{aa}(t)$  and  $c_a(t)$  can then be calculated from the evolution of the mode operators,

$$i \begin{pmatrix} \dot{A}_+ \\ \dot{A}_- \end{pmatrix} = \begin{pmatrix} \omega_+(t) - i\kappa/2 & -\dot{\theta}(t) \\ -\dot{\theta}(t) & \omega_-(t) - i\kappa/2 \end{pmatrix} \begin{pmatrix} A_+ \\ A_- \end{pmatrix} + g_0 \mathcal{N}(t) \begin{pmatrix} e^{-i\theta(t)} \\ e^{-i\theta(t)} \end{pmatrix}. \quad (44)$$

From this expression we identify three potential effects of phase noise. First, intensity fluctuations  $\sim \mathcal{N}(t)$  introduce a random displacement  $c_a(t)$  which is related to the additional heating discussed above. Second, the fluctuating cavity amplitude  $|\alpha(t)|$  leads to a fluctuating normal mode splitting,  $\omega_\pm(t) \simeq \omega_\pm \pm \delta\omega(t)$  where  $\delta\omega(t) \simeq g_0 \mathcal{N}(t)/|\alpha_0|$ . Third, phase fluctuations of the intra-cavity field causes non-adiabatic transitions between the modes  $A_\pm$  proportional to  $\dot{\theta}(t)$ .

### High frequency noise

Based on our discussions so far we expect that intensity fluctuations are the dominant decoherence mechanism. In a first approximation we therefore neglect the time dependence of  $\omega(t)$  and  $\theta(t)$  and study the effect of  $\mathcal{N}(t)$  only. By integrating Eq. (44) we then obtain

$$\mathcal{F}(t) = \mathcal{F}_0 \times e^{-D(t)}, \quad (45)$$

where

$$D(t) = \{|c_a(t)|^2\}_{cl} = 2g_0^2 \int_0^t d\tau \int_0^\tau d\tau' e^{i\omega_m(\tau-\tau')} e^{-\kappa(\tau+\tau')/2} \times \sin(|G|\tau) \sin(|G|\tau') \{\mathcal{N}(\tau)\mathcal{N}(\tau')\}_{cl}. \quad (46)$$

For a full oscillation period  $t = \pi/(|G|)$  and  $|G| \gg \kappa$  this expression reduces to

$$D(t = \pi/(|G|)) \approx 8|\alpha_0|^2 \frac{S_\phi(\omega_m)}{\kappa}. \quad (47)$$

As a consequence the error  $\epsilon = 1 - \mathcal{F}$  for a full oscillation between cavity and resonator mode is

$$\epsilon \approx \frac{\pi\kappa}{g_0|\alpha_0|} + 8|\alpha_0|^2 \frac{S_\phi(\omega_m)}{\kappa}. \quad (48)$$

For fixed cavity parameters there is an optimal field amplitude  $|\alpha_0|$  for which, apart from a numerical prefactor, the error scales as

$$\epsilon \approx \sqrt[3]{\kappa S_\phi(\omega_m)/g_0^2}. \quad (49)$$

We see that apart from other imperfections, achieving  $S_\phi(\omega_m) \ll g_0^2/\kappa$  is a necessary requirement for the observation of coherent dynamics in opto-mechanical systems. Since  $\kappa$  must also exceed the mechanical heating rate  $\Gamma_m$  this result implies that the observation of coherent oscillations puts more stringent bounds on the acceptable level of phase noise than just cooling.

#### Low frequency noise

While due to the scaling  $\sim |\alpha_0|^2$  intensity fluctuations impose a severe limitation on coherent oscillations, this effect depends on phase noise at relatively high frequencies  $\omega \sim \omega_m$ . Since the relevant system dynamics occurs on a slower timescale  $|G|^{-1}$  this noise can in principle be filtered out in a carefully designed experimental setup. An important question therefore is whether or not other decoherence mechanisms exist which depend on low frequency regime of the phase noise spectrum.

To address this question we now assume that the phase noise spectrum  $S_\phi(\Omega)$  has relevant contributions only at  $\Omega \ll \omega_m$ . This assumption allows us to omit the term  $\sim \mathcal{N}(t)$  in Eq. (44) and to study the effects of  $\delta\omega(t)$  and  $\dot{\theta}(t)$  only. For a simplified discussion we will also neglect cross-correlations between these two stochastic quantities. In App. C we show that the resulting fidelity is then given by

$$\mathcal{F}(t) \approx \exp\left(-\{|1 + \cos(|G|t)e^{-\kappa t/2}e^{-W(t)/2}e^{-R(t)/2}\}_{cl}\right). \quad (50)$$

Compared to the case of a noiseless laser we obtain two additional contributions to decoherence. Here

$$W(t) = 2 \int_0^t ds \int_0^s ds' \{\delta\omega(s)\delta\omega(s')\}_{cl}, \quad (51)$$

describes dephasing due to a modulated normal mode splitting and

$$R(t) = 2\text{Re} \int_0^t ds \int_0^s ds' e^{iG(s-s')} \{\dot{\theta}(s)\dot{\theta}(s')\}_{cl}, \quad (52)$$

describes non-adiabatic transitions between the normal modes.

As anticipated above Eq. (51) and Eq. (52) differ qualitatively from the heating discussed above in the sense that the quantities  $W(t)$  and  $R(t)$  depend on noise far below the resonator frequency. To see this more explicitly we use  $\delta\omega(t) = g_0\mathcal{N}(t)/|\alpha_0|$  and obtain

$$W(t) \approx \frac{g_0^2|\alpha_0|^2}{\omega_m^2} \int_{|\Omega| < \omega_m} \frac{d\Omega}{2\pi} \frac{4\sin^2(\Omega t/2)}{\Omega^2} S_\phi(\Omega). \quad (53)$$

An upper bound for this integral can be obtained from the long time limit,

$$W(t) \leq \frac{|G|^2}{\omega_m^2} S_\phi(0) \times t. \quad (54)$$

We see a dependence on zero frequency noise  $S_\phi(0) \approx \Gamma_L$ , but since this noise is far detuned from the cavity resonance this dephasing process is suppressed by  $|G|^2/\omega_m^2$ . For the parameter regime of interest the resulting error for a coherent oscillation  $\epsilon \approx |G|\Gamma_L/\omega_m^2 \ll 1$  is therefore always smaller than the error due to photon loss.

A similar conclusion can be obtained for decoherence caused by non-adiabatic transitions. A simple estimate for the upper bound on  $R(t)$  can be obtained by assuming that the intracavity phase  $2\theta(t)$  follows the laser phase  $\phi(t)$  adiabatically (see Eq. (15)). Under this approximation we obtain

$$R(t) \leq \frac{1}{4} S_\phi(|G|) \times t. \quad (55)$$

Again we see that for  $\Gamma_L \ll \kappa$  this decoherence process is negligible compared to cavity loss and a more accurate calculation predicts a further reduction due to a suppression of low frequency noise by the cavity response. We conclude that while low frequency noise does affect the dynamics of the opto-mechanical system it is for the parameter regime of interest negligible compared to photon loss.

#### IV. SUMMARY & CONCLUSIONS

In summary we have analyzed the effect of laser phase noise on cooling and coherent dynamics in a generic opto-mechanical system comprised of a driven optical cavity with a vibrating end-mirror. Our approach significantly extends and generalizes previous treatments to this problem and provides a rigorous way to study different aspects of the laser noise in such systems. For opto-mechanical cooling in the weak coupling regime our predictions for a final occupation number  $n_0 \sim \mathcal{O}(10)$  are consistent with experimental data and we derive a condition for the noise spectrum which

is required in future experiments to achieve ground state cooling. We have also show that the discrepancy between experiments and previous theoretical calculations [19] is based on the assumption of a white noise model, which does not lead to physically meaningful predictions. For opto-mechanical systems in the strong coupling regime we have shown that the effects of phase noise still scale linearly with the intensity but the observation of coherent oscillations places more stringent bounds on tolerable level of phase noise. Nevertheless, we conclude that ground state cooling and coherent state transfer experiments can be achieved with state of the art laser stabilization techniques.

*Acknowledgments* The authors thank L. Diosi, A. Schliesser, D. Vitali, J. Ye and P. Zoller for stimulating discussions. P. R. acknowledges support by the NSF through a grant for ITAMP. C.G. is thankful for support from Euroquam Austrian Science Fund project I1 19 N16 CMMC and K. H., C. G. and M.A. announce support by the Austrian Science Foundation under SFB FOQUS.

### APPENDIX A: EVALUATION OF THE AMPLITUDE FLUCTUATION SPECTRUM

We evaluate the amplitude fluctuation spectrum  $S_A(\omega)$  defined in Eq. (12) in the limit  $\Gamma_L/\kappa \ll 1$ . From the definition of  $\alpha(t)$  given in Eq. (8) we find that under stationary conditions the correlation function of the classical cavity field can be written in the form

$$\{\alpha^*(\tau)\alpha(0)\}_{cl} = \int_0^\infty dy e^{-2\kappa y} \int_{-2y}^{2y} dx e^{-i\Delta x} C_2(\tau - x), \quad (A1)$$

where  $C_2(t) = \{\mathcal{E}^*(t)\mathcal{E}(0)\}_{cl}$  is the two point correlation function of the driving field. For Gaussian phase noise we obtain

$$C_2(t) = |\mathcal{E}_0|^2 e^{-\frac{1}{2}\langle\phi^2(t)\rangle_{cl}}, \quad (A2)$$

where  $\phi(t) = \int_{-t/2}^{t/2} ds \dot{\phi}(s)$ . Using the noise model defined in Eq. (6) we see that the expectation value in the exponent is bound by  $\langle\phi^2(t)\rangle_{cl} \leq \Gamma_L \gamma_c t^2$  for  $t \lesssim \gamma_c^{-1}$  and  $\langle\phi^2(t)\rangle_{cl} \leq \Gamma_L t$  for long times. Therefore, in the limit  $\Gamma_L \ll \kappa$  we can expand the exponential in Eq. (A2) to first order,

$$C_2(t) \simeq |\mathcal{E}_0|^2 \left( 1 - \frac{1}{2} \int_{-t/2}^{t/2} ds \int_{-t/2}^{t/2} ds' \{\dot{\phi}(s)\dot{\phi}(s')\}_{cl} \right). \quad (A3)$$

Equivalently, we can rewrite this expression in terms of the noise spectrum,

$$C_2(t) \simeq |\mathcal{E}_0|^2 \left( 1 - \int \frac{d\Omega}{2\pi} S_{\dot{\phi}}(\Omega) \frac{1 - \cos(\Omega t)}{\Omega^2} \right). \quad (A4)$$

We insert this result back into the definition of  $\{\alpha^*(\tau)\alpha(0)\}_{cl}$  and  $S_A(\omega)$  and evaluate the remaining integrals. Since the resulting general expressions are lengthy we here only present the results for the sideband resolved regime  $\kappa \ll \omega_m$ , where  $\Delta_{op} = -\omega_m$  and  $W_0 = 2g_0^2|\alpha_0|^2/\kappa$ . For the red sideband we obtain

$$\frac{S_A(\omega_m)}{W_0} \simeq 1 - \int \frac{d\Omega}{2\pi} \frac{S_{\dot{\phi}}(\Omega)}{\kappa^2 + \Omega^2}. \quad (A5)$$

Using the noise model defined in Eq. (6) we see that in the white noise limit  $\gamma_c \gg \kappa$  corrections are of the order of  $S_{\dot{\phi}}(0)/(2\kappa) \approx \Gamma_L/\kappa$  while in the opposite limit they scale as  $\Gamma_L \gamma_c/\kappa^2$ . For the blue sideband transitions we obtain the result presented in Eq. (18).

### APPENDIX B: EVALUATION OF THE INTENSITY FLUCTUATION SPECTRUM

We evaluate the intensity fluctuation correlation function  $C_N(\tau) = \{N(\tau)N(0)\}_{cl}$  in the limit  $\Gamma_L \ll \kappa$ . Ignoring small corrections of order  $\mathcal{O}(g_0/\kappa)$  the average cavity photon number is  $\bar{n}_{ph} = \{\langle a^\dagger a \rangle_q\}_{cl} \simeq \{|\alpha(t)|^2\}_{cl}$  and  $N(t) = |\alpha(t)|^2 - \{|\alpha(t)|^2\}_{cl}$ . Under stationary conditions  $C_N(\tau)$  can be written as

$$C_N(\tau) = \int_0^\infty dy_1 dy_2 e^{-2\kappa y_1} e^{-2\kappa y_2} \times \int_{-2y_1}^{2y_1} dx_1 \int_{-2y_2}^{2y_2} dx_2 e^{-i\Delta x_1} e^{-i\Delta x_2} C_4(\tau - (y_1 - y_2), x_1, x_2). \quad (B1)$$

Here we have introduced the four point field correlation function

$$C_4(T, t_1, t_2) = \{\mathcal{E}^*(T + t_1/2) \mathcal{E}(T - t_1/2) \mathcal{E}^*(t_2/2) \mathcal{E}(-t_2/2)\}_{cl} - \{\mathcal{E}^*(t_1/2) \mathcal{E}(-t_1/2)\} \{\mathcal{E}^*(t_2/2) \mathcal{E}(-t_2/2)\}_{cl}. \quad (B2)$$

For phase noise with Gaussian statistics the four point correlation function is given by

$$C_4(T, t_1, t_2) = |\mathcal{E}_0|^4 \left( e^{-\frac{1}{2}\langle\Phi^2(T, t_1, t_2)\rangle_{cl}} - e^{-\frac{1}{2}\langle\phi^2(t_1) + \phi^2(t_2)\rangle_{cl}} \right), \quad (B3)$$

where  $\Phi(\tau, t_1, t_2) = \int_{\tau-t_1/2}^{\tau+t_1/2} ds \dot{\phi}(s) + \int_{-t_2/2}^{+t_2/2} ds \dot{\phi}(s)$  and  $\phi(t) = \int_{-t/2}^{+t/2} ds \dot{\phi}(s)$ . Following the same argumentation as in the evaluation of  $S_A(\omega)$  in App. A we can in the limit  $\Gamma_L \ll \kappa$  expand the exponentials in Eq. (B3) to first order,

$$C_4(T, t_1, t_2) \simeq -|\mathcal{E}_0|^4 \int_{\tau-t_1}^{\tau+t_1} ds \int_{-t_2}^{+t_2} ds' \{\dot{\phi}(s)\dot{\phi}(s')\}_{cl}, \quad (B4)$$

or in terms of the noise spectrum  $S_{\dot{\phi}}(\Omega)$ ,

$$C_4(T, t_1, t_2) \simeq -4|\mathcal{E}_0|^4 \int \frac{d\Omega}{2\pi} S_{\dot{\phi}}(\Omega) \frac{\sin(\Omega t_1/2) \sin(\Omega t_2/2)}{\Omega^2} e^{-i\Omega T}. \quad (B5)$$

After inserting this expression back into Eq. (B1) and evaluating the remaining integrals we obtain

$$C_N(\tau) \simeq |\alpha_0|^4 \int \frac{d\Omega}{2\pi} \frac{4\Delta^2 S_{\dot{\phi}}(\Omega) e^{-i\Omega \tau}}{(\Delta^4 + 2\Delta^2(\kappa^2 - \Omega^2) + (\kappa^2 + \Omega^2)^2)}. \quad (B6)$$

For the evaluation of  $S_N(\omega_m)$  defined in Eq. (13) the integral over  $\tau$  results in a term  $\sim \delta(\Omega - \omega_m)$  and we end up with

$$S_N(\omega_m) = g_0^2 |\alpha_0|^4 \frac{4\Delta^2 S_{\dot{\phi}}(\omega_m)}{(\Delta^4 + 2\Delta^2(\kappa^2 - \omega_m^2) + (\kappa^2 + \omega_m^2)^2)}. \quad (B7)$$



For the optimal detuning  $\Delta = -\sqrt{\omega_m^2 + \kappa^2}$  this expression simplifies to  $S_N(\omega_m) = g_0^2 |\alpha_0|^4 S_{\dot{\phi}(\omega_m)}/\kappa^2$ .

In the strong coupling regime heating rates depend on the fluctuation spectrum evaluated at the imaginary frequency,  $S_N(\omega_{\pm} + i\kappa/2)$ . In this case we obtain

$$S_N(\omega_{\pm} + i\kappa/2) = g_0^2 |\alpha_0|^4 \int \frac{d\Omega}{2\pi} S_{\dot{\phi}}(\Omega) \frac{4\kappa}{\kappa^2 + 4(\Omega - \omega_{\pm})^2} \times \frac{4\Delta^2}{(\Delta^4 + 2\Delta^2(\kappa^2 - \Omega^2) + (\kappa^2 + \Omega^2)^2)}. \quad (\text{B8})$$

For  $\Delta = -\omega_m$  and  $\kappa \ll G$  this integral is dominated by two contributions from the resonances at  $\Omega = \omega_m$  and  $\Omega = \omega_{\pm}$ . In the limit  $\kappa \rightarrow 0$  we obtain the result presented in Eq. (37).

### APPENDIX C: LOW FREQUENCY NOISE

In the absence of intensity fluctuations,  $\mathcal{N}(t) = 0$ , the formal solution of Eq. (44) is given by

$$\begin{pmatrix} A_+(t) \\ A_-(t) \end{pmatrix} = e^{-i\mathbf{M}_0 t} \left( \mathcal{T} e^{-i \int_0^t \mathbf{M}(s) ds} \right) \begin{pmatrix} A_+(0) \\ A_-(0) \end{pmatrix}, \quad (\text{C1})$$

where  $\mathcal{T}$  is the time ordering operator,

$$\mathbf{M}_0 = \begin{pmatrix} \omega_+ - i\kappa/2 & 0 \\ 0 & \omega_- - i\kappa/2 \end{pmatrix}, \quad (\text{C2})$$

and

$$\mathbf{M}(t) = e^{i\mathbf{M}_0 t} \begin{pmatrix} \delta\omega(t) & -\dot{\theta}(t) \\ -\dot{\theta}(t) & -\delta\omega(t) \end{pmatrix} e^{-i\mathbf{M}_0 t}. \quad (\text{C3})$$

To obtain  $\{c_{aa}(t)\}_{cl}$  we take the classical average of Eq. (C1) and evaluate the average of the exponential of  $\mathbf{M}(t)$  using a second order cumulant expansion. Then

$$\begin{pmatrix} \{A_+(t)\}_{cl} \\ \{A_-(t)\}_{cl} \end{pmatrix} = e^{-i\mathbf{M}_0 t} e^{-\bar{\mathbf{M}}(t)} \begin{pmatrix} A_+(0) \\ A_-(0) \end{pmatrix}, \quad (\text{C4})$$

where

$$\bar{\mathbf{M}}(t) = \frac{1}{2} \begin{pmatrix} W(t) + R(t) + iI(t) & -X^*(t) \\ X(t) & W(t) + R(t) - iI(t) \end{pmatrix}. \quad (\text{C5})$$

Here  $W(t)$  and  $R(t)$  are defined in Eq. (51), and Eq. (52), and  $I(t)$  is the imaginary part of integral in Eq. (52) which leads to a small shift of the oscillation frequency. Finally,

$$X(t) = 2 \int_0^t ds \int_0^s ds' \left( e^{-iG|s'|} \{\delta\omega(s)\dot{\theta}(s')\}_{cl} - e^{-iG|s|} \{\delta\omega(s')\dot{\theta}(s)\}_{cl} \right), \quad (\text{C6})$$

is an additional cross term which arises from correlations between  $\delta\omega(t)$  and  $\dot{\theta}(t)$ . However, using the a rough approximation for  $\alpha(t) \approx \alpha_0(1 - i\dot{\phi}(t)/(\kappa + i\omega_m))$  and  $\theta(t) \approx \phi(t)$  we obtain  $\{\delta\omega(t)\dot{\theta}(t')\}_{cl} \approx |G|/\omega_m \{\dot{\phi}(t)\dot{\phi}(t')\}_{cl}$  and also this term does not contribute significantly to decoherence. For a simplified discussion, both the cross term and the frequency shift are omitted in Eq. (50).

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